26　定積分

基本問題 & 解法のポイント

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(1)

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)dx$$

$$= \int_0^{\frac{\pi}{2}} (\cos x - \cos x \sin^2 x)dx$$

$$= \left[\sin x - \frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3}$$

(2)

$$\int_0^{\pi} x \sin x dx = [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x dx$$

$$= \pi$$

(3)

$$\int_0^{2\sqrt{2}} \frac{dx}{8+x^2} = \frac{1}{8}\int_0^{2\sqrt{2}} \frac{dx}{1+\left(\frac{x}{2\sqrt{2}}\right)^2}$$

$\frac{x}{2\sqrt{2}} = \tan\theta \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$とおくと，$dx = \frac{2\sqrt{2}d\theta}{\cos^2\theta}$，$x = 2\sqrt{2} \Leftrightarrow \theta = \frac{\pi}{4}$，$x = 0 \Leftrightarrow \theta = 0$より，

$$\int_0^{2\sqrt{2}} \frac{dx}{8+x^2} = \frac{1}{8}\int_0^{\frac{\pi}{4}} \frac{\frac{2\sqrt{2}d\theta}{\cos^2\theta}}{1+\tan^2\theta}$$

$$= \frac{\sqrt{2}}{4}\int_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{\sqrt{2}}{16}\pi$$

(4)

$$\int_1^e x^2(\log x - 1)dx = \left[\frac{x^3}{3}(\log x - 1)\right]_1^e - \frac{1}{3}\int_1^e x^3 \cdot \frac{1}{x} dx$$

$$= \frac{1}{3} - \frac{1}{9}\left[x^3\right]_1^e$$

$$= \frac{1}{9}(4 - e^3)$$

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(1)

$$\int_0^{\pi}|\sin x+\cos x|dx=\int_0^{\pi}\left|\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right|dx$$
$$=\sqrt{2}\left\{\int_0^{\frac{3}{4}\pi}\sin\left(x+\frac{\pi}{4}\right)dx-\int_{\frac{3}{4}\pi}^{\pi}\sin\left(x+\frac{\pi}{4}\right)dx\right\}$$
$$=\sqrt{2}\left\{\int_0^{\frac{3}{4}\pi}\sin\left(x+\frac{\pi}{4}\right)dx+\int_{\pi}^{\frac{3}{4}\pi}\sin\left(x+\frac{\pi}{4}\right)dx\right\}$$
$$=-\sqrt{2}\left\{\left[\cos\left(x+\frac{\pi}{4}\right)\right]_0^{\frac{3}{4}\pi}+\left[\cos\left(x+\frac{\pi}{4}\right)\right]_{\pi}^{\frac{3}{4}\pi}\right\}$$
$$=\sqrt{2}\left\{\left[\cos\left(x+\frac{\pi}{4}\right)\right]_{\frac{3}{4}\pi}^{0}+\left[\cos\left(x+\frac{\pi}{4}\right)\right]_{\frac{3}{4}\pi}^{\pi}\right\}$$
$$=\sqrt{2}\left(\frac{\sqrt{2}}{2}+1-\frac{\sqrt{2}}{2}+1\right)$$
$$=2\sqrt{2}$$

(2)

$$\int_0^{2\pi}x^2|\sin x|dx=\int_0^{\pi}x^2\sin xdx-\int_{\pi}^{2\pi}x^2\sin xdx$$
$$=\int_0^{\pi}x^2\sin xdx+\int_{2\pi}^{\pi}x^2\sin xdx$$

これと,

$$\int x^2\sin xdx=-x^2\cos x+2\int x\cos xdx$$
$$=-x^2\cos x+2\left(x\sin x-\int\sin xdx\right)$$
$$=-x^2\cos x+2x\sin x+2\cos x+C$$

より,

$$\int_0^{2\pi}x^2|\sin x|dx=\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi}+\left[-x^2\cos x+2x\sin x+2\cos x\right]_{2\pi}^{\pi}$$
$$=2\left(\pi^2-2\right)-2-\left(-4\pi^2+2\right)$$
$$=6\pi^2-8$$

A

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(1)

解法 **1**

$\log x = t$ とおくと，$x = e^t$，$\dfrac{1}{x} = \dfrac{dt}{dx}$ より，$dx = e^t dt$

また，$x = e \Leftrightarrow t = 1,\ x = 1 \Leftrightarrow t = 0$

よって，$\displaystyle\int_1^e 5^{\log x} dx = \int_0^1 5^t e^t dt$ ・・・①

$$\int_0^1 5^t e^t dt = \left[5^t e^t\right]_0^1 - \int_0^1 \left(5^t\right)' e^t dt$$

$$= 5e - 1 - \log 5 \int_0^1 5^t e^t dt$$

より，$\displaystyle(1 + \log 5)\int_0^1 5^t e^t dt = 5e - 1$ ・・・②

①，②より，$\displaystyle\int_1^e 5^{\log x} dx = \frac{5e - 1}{1 + \log 5}$

補足

$\left(5^t\right)' = \left(e^{\log 5^t}\right)' = \left(e^{t \log 5}\right)' = \log 5 e^{t \log 5} = \log 5 \cdot 5^t$ より，

解法 **2**

$$\int_1^e 5^{\log x} dx = \int_0^1 5^t e^t dt$$

$$= \int_0^1 e^{t \log 5} e^t dt$$

$$= \int_0^1 e^{t(\log 5 + 1)} dt$$

$$= \int_0^1 e^{t \log 5e} dt$$

$$= \left[\frac{e^{t \log 5et}}{\log 5e}\right]_0^1$$

$$= \frac{e^{\log 5e} - 1}{\log 5e}$$

$$= \frac{5e - 1}{\log 5 + \log e}$$

$$= \frac{5e - 1}{\log 5 + 1}$$

(2)

$x=\tan\theta\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$ とおくと， $x=1\Leftrightarrow\theta=\frac{\pi}{4},\ \theta=0\Leftrightarrow\theta=0$

$$\begin{aligned}\frac{x+1}{(x^2+1)}dx&=\frac{\tan\theta+1}{(\tan^2\theta+1)^2}\frac{1}{\cos^2\theta}d\theta\\&=\frac{\tan\theta+1}{\left(\frac{1}{\cos^2\theta}\right)^2\cos^2\theta}d\theta\\&=\cos^2\theta(\tan\theta+1)d\theta\\&=\cos^2\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)d\theta\\&=(\cos\theta\sin\theta+\cos^2\theta)d\theta\\&=\left(\frac{\sin2\theta}{2}+\frac{\cos2\theta+1}{2}\right)d\theta\\&=\frac{1}{4}(2\sin2\theta+2\cos2\theta+2)d\theta\end{aligned}$$

$$\begin{aligned}\therefore\int_0^1\frac{x+1}{(x^2+1)^2}dx&=\frac{1}{4}\int_0^{\frac{\pi}{4}}(2\sin2\theta+2\cos2\theta+2)d\theta\\&=\frac{1}{4}\left[-\cos2\theta+\sin2\theta+2\theta\right]_0^{\frac{\pi}{4}}\\&=\frac{1}{4}\left(1+\frac{\pi}{2}+1\right)\\&=\frac{1}{2}+\frac{\pi}{8}\end{aligned}$$

(3)

解法 1

$$\begin{aligned}\int_1^e x(\log x)^2dx&=\left[\frac{x^2(\log x)^2}{2}\right]_1^e-\int_1^e\frac{x^2\cdot\frac{2}{x}\log x}{2}dx\\&=\frac{e^2}{2}-\int_1^e x\log xdx\\&=\frac{e^2}{2}-\left[\frac{x^2\log x}{2}\right]_0^1+\int_1^e\frac{x}{2}dx\\&=\frac{e^2}{2}-\frac{e^2}{2}+\left[\frac{x^2}{4}\right]_1^e\\&=\frac{e^2-1}{4}\end{aligned}$$

解法 2

$\log x = t$ とおくと， $x = e \Leftrightarrow t = 1,\ x = 1 \Leftrightarrow t = 0$ より，

$$\begin{aligned}\int_1^e x(\log x)^2\,dx &= \int_0^1 e^t t^2 e^t\,dt\\ &= \int_0^1 t^2 e^{2t}\,dt\\ &= \left[\frac{t^2 e^{2t}}{2}\right]_0^1 - \int_0^1 te^{2t}\,dt\\ &= \frac{e^2}{2} - \left[\frac{te^{2t}}{2}\right]_0^1 + \int_0^1 \frac{e^{2t}}{2}\,dt\\ &= \frac{e^2}{2} - \frac{e^2}{2} + \left[\frac{e^{2t}}{4}\right]_0^1\\ &= \frac{e^2-1}{4}\end{aligned}$$

(4)

$$\begin{aligned}\frac{dx}{\sin^2 x + 3\cos^2 x} &= \frac{1}{\tan^2 x + 3}\cdot\frac{dx}{\cos^2 x}\\ &= \frac{1}{3\left\{\left(\frac{\tan x}{\sqrt{3}}\right)^2 + 1\right\}}\cdot\frac{dx}{\cos^2 x}\end{aligned}$$

ここで， $\dfrac{\tan x}{\sqrt{3}} = \tan\theta$ とおくと， $\dfrac{dx}{\sqrt{3}\cos^2 x} = \dfrac{d\theta}{\cos^2\theta}$

$$\begin{aligned}\frac{dx}{\sin^2 x + 3\cos^2 x} &= \frac{1}{3\left\{\left(\frac{\tan x}{\sqrt{3}}\right)^2 + 1\right\}}\cdot\frac{dx}{\cos^2 x}\\ &= \frac{1}{3(\tan^2\theta + 1)}\cdot\frac{\sqrt{3}d\theta}{\cos^2\theta}\\ &= \frac{\cos^2\theta}{3}\cdot\frac{\sqrt{3}}{\cos^2\theta}d\theta\\ &= \frac{\sqrt{3}}{3}d\theta\end{aligned}$$

これと $x = \dfrac{\pi}{4} \Leftrightarrow \theta = \dfrac{\pi}{6},\ x = 0 \Leftrightarrow \theta = 0$ より，

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^2 x + 3\cos^2 x} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}}{3}\,d\theta = \frac{\sqrt{3}}{18}\pi$$

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$$\frac{dt}{dx}=\frac{d}{dx}\tan\frac{x}{2}$$
$$=\frac{1}{2\cos^2\frac{x}{2}}$$
$$=\frac{1+\tan^2\frac{x}{2}}{2}$$
$$=\frac{1+t^2}{2}$$

$$\therefore dx=\frac{2}{1+t^2}dt \quad ・・・①$$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$
$$=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$
$$=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\therefore \sin x=\frac{2t}{1+t^2} \quad ・・・②$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$$
$$=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$
$$=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\therefore \cos x=\frac{1-t^2}{1+t^2} \quad ・・・③$$

①～③および $x=\frac{\pi}{2}\Leftrightarrow t=1,\ x=0\Leftrightarrow t=0$ より，

$$\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x+\cos x}dx=\int_0^1\frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$= \int_0^1 \frac{1}{1+t} dt$$
$$= \left[\log|1+t| \right]_0^1$$
$$= \log 2$$

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$(n-1)\pi \leq x \leq n\pi$ のとき $|\sin x| = (-1)^{n+1} \sin x$

よって，$\displaystyle\int_{(n-1)\pi}^{n\pi} e^{-x} |\sin x| dx = (-1)^{n+1} \int_{(n-1)\pi}^{n\pi} e^{-x} \sin x dx$

ここで，

$$\begin{aligned}\int_{n-1}^{n\pi} e^{-x} \sin x dx &= \left[-e^{-x} \sin x \right]_{(n-1)\pi}^{n\pi} + \int_{(n-1)\pi}^{n\pi} e^{-x} \cos x dx \\ &= \left[-e^{-x} \cos x \right]_{(n-1)\pi}^{n\pi} - \int_{(n-1)\pi}^{n\pi} e^{-x} \sin x dx \\ &= -e^{-n\pi} \cos n\pi + e^{-(n-1)\pi} \cos (n-1)\pi - \int_{(n-1)\pi}^{n\pi} e^{-x} \sin x dx\end{aligned}$$

より，

$$\begin{aligned}\int_{n-1}^{n\pi} e^{-x} \sin x dx &= \frac{-e^{-n\pi} \cos n\pi + e^{-(n-1)\pi} \cos (n-1)\pi}{2} \\ &= \frac{-(-1)^n e^{-n\pi} + (-1)^{n-1} e^{-(n-1)\pi}}{2} \\ &= \frac{(-1)^{n-1} \left(e^{n\pi} + e^{-(n-1)\pi} \right)}{2}\end{aligned}$$

ゆえに，

$$\begin{aligned}\int_{(n-1)\pi}^{n\pi} e^{-x} |\sin x| dx &= (-1)^{n+1} \cdot \frac{(-1)^{n-1} \left(e^{n\pi} + e^{-(n-1)\pi} \right)}{2} \\ &= \frac{e^{-n\pi} \left(1 + e^{-\pi} \right)}{2} \\ &= \frac{1 + e^{\pi}}{2e^{n\pi}}\end{aligned}$$

補足

$\left(e^{-x} \sin x \right)' = -e^{-x} \sin x + e^{-x} \cos x$，$\left(e^{-x} \cos x \right)' = -e^{-x} \sin x - e^{-x} \cos x$ より，

$e^{-x} \sin x = -\dfrac{\left(e^{-x} \sin x \right)' + \left(e^{-x} \cos x \right)'}{2}$ を利用する方法もある。

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$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\frac{\pi}{2}} f(x)dx$ の証明

解法 1

$\frac{\pi}{2}-x=t$ とおくと,

$$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx = \int_{\frac{\pi}{2}}^{0} f(t)\cdot(-dt)$$
$$= \int_0^{\frac{\pi}{2}} f(t)dt$$
$$= \int_0^{\frac{\pi}{2}} f(x)dx$$

解法 2

$y=f(x)$ 上の点 $(x,\ y)$ を $x=\frac{\pi}{4}$ 関して対称移動した点を $(X,\ Y)$ とすると,

$\frac{x+X}{2}=\frac{\pi}{4}$ より, $x=\frac{\pi}{2}-X$

また, $y=Y$

これらを $y=f(x)$ に代入すると, $Y=f\left(\frac{\pi}{2}-X\right)$

よって, $y=f(x)$ と $y=f\left(\frac{\pi}{2}-x\right)$ は $x=\frac{\pi}{4}$ 関して対称である。

これと, 区間 $\left[0,\ \frac{\pi}{2}\right]$ の中線が $x=\frac{\pi}{4}$ で, $f(x)$ がその区間で連続であることから

$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\frac{\pi}{2}} f(x)dx$ が成り立つ。

$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x+\cos x}dx$ の値

$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\frac{\pi}{2}} f(x)dx$ より,

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\sin 3\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\cos x+\sin x}dx$$

よって，

$$
\begin{aligned}
2\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}dx &= \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}dx + \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\cos x + \sin x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin 3x - \cos 3x}{\sin x + \cos x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{3\sin x - 4\sin^3 x - \left(-3\cos x + 4\cos^3 x\right)}{\sin x + \cos x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{3(\sin x + \cos x) - 4\left(\sin^3 x + \cos^3 x\right)}{\sin x + \cos x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{3(\sin x + \cos x) - 4(\sin x + \cos x)\left(\sin^2 x - \sin x\cos x + \cos^2 x\right)}{\sin x + \cos x}dx \\
&= \int_0^{\frac{\pi}{2}} \{3 - 4(1 - \sin x\cos x)\}dx \\
&= \int_0^{\frac{\pi}{2}} (2\sin 2x - 1)dx \\
&= \left[-\cos 2x - x\right]_0^{\frac{\pi}{2}} \\
&= 2 - \frac{\pi}{2}
\end{aligned}
$$

ゆえに，$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}dx = 1 - \frac{\pi}{4}$

B

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(1)

$$\int_{-\pi}^{\pi} x\sin x dx = [-x\cos x]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x dx$$
$$= 2\pi + [\sin x]_{-\pi}^{\pi}$$
$$= 2\pi$$

(2)

$$\int_{-\pi}^{\pi} \sin 2x \sin 3x dx = \int_{-\pi}^{\pi} (\cos x - \cos 5x) dx$$
$$= \left[\sin x - \frac{\sin 5x}{5}\right]_{-\pi}^{\pi}$$
$$= 0$$

(3)

$m = n$ のとき

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \sin^2 mx dx$$
$$= \int_{-\pi}^{\pi} \frac{1-\cos 2mx}{2} dx$$
$$= \left[\frac{x}{2} - \frac{\sin 2mx}{4}\right]_{-\pi}^{\pi}$$
$$= \pi$$

$m \neq n$ のとき

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(m-n)x - \cos(m+n)x\} dx$$
$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}$$
$$= 0$$

(4)

(3)より,

$$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{2013} \sin kx\right)^2 dx = \int_{-\pi}^{\pi} \sum_{k=1}^{2013} \sin^2 kx dx$$
$$= 2013\pi$$

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(1)

題意が成り立つことを数学的帰納法により証明する。

$\cos n\theta = T_n(\cos\theta)$　・・・①　とすると，

(i)　$n=0$ のとき

$T_0(\cos\theta)=1$，$1=\cos(0\cdot\theta)$ より，$\cos(0\cdot\theta)=T_0(\cos\theta)$　よって，①が成り立つ。

(ii)　$n=1$ のとき

$T_1(\cos\theta)=\cos\theta$，$\cos\theta=\cos(1\cdot\theta)$ より，$\cos(1\cdot\theta)=T_1(\cos\theta)$　よって，①が成り立つ。

(iii)　$n=k,\ k+1\ (k=0,\ 1,\ 2,\ \cdots)$ で①が成り立つと仮定する。

$$\begin{aligned}
T_{k+2}(\cos\theta) &= 2\cos\theta\, T_{k+1}(\cos\theta)-T_k(\cos\theta)\\
&= 2\cos\theta\cdot\cos(k+1)\theta-\cos k\theta\\
&= 2\cos\theta\cdot\cos(k+1)\theta-\cos\{(k+1)\theta-\theta\}\\
&= 2\cos\theta\cdot\cos(k+1)\theta-\{\cos(k+1)\theta\cdot\cos\theta+\sin(k+1)\theta\cdot\sin\theta\}\\
&= \cos(k+1)\theta\cdot\cos\theta-\sin(k+1)\theta\cdot\sin\theta\\
&= \cos\{(k+1)\theta+\theta\}\\
&= \cos(k+2)\theta
\end{aligned}$$

より，$n=k+2$ のときも①が成り立つ。

(i)，(ii)，(iii)より，題意が成り立つ。

(2)

$x=\cos\theta$ とおくと，

$$\begin{aligned}
\int_{-1}^{1} T_n(x)dx &= \int_{\pi}^{0} T_n(\cos\theta)(-\sin\theta)d\theta\\
&= \int_0^{\pi}\cos n\theta\sin\theta\, d\theta\\
&= \frac{1}{2}\int_0^{\pi}\{\sin(n\theta+\theta)-\sin(n\theta-\theta)\}d\theta\\
&= \frac{1}{2}\int_0^{\pi}\{\sin(n+1)\theta-\sin(n-1)\theta\}d\theta\\
&= \frac{1}{2}\left[\frac{\cos(n-1)\theta}{n-1}-\frac{\cos(n+1)\theta}{n+1}\right]_0^{\pi}\\
&= \frac{1}{2}\left(\frac{\cos(n-1)\pi}{n-1}-\frac{\cos(n+1)\pi}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}\right)
\end{aligned}$$

よって，

n が偶数ならば $\displaystyle\int_{-1}^{1} T_n(x)dx=\frac{1}{2}\left(\frac{-1}{n-1}-\frac{-1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}\right)=\frac{1}{n+1}-\frac{1}{n-1}$

n が奇数ならば $\displaystyle\int_{-1}^{1} T_n(x)dx=\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}\right)=0$

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(1)

$y=\dfrac{e^x}{e^x+1}$ より，$e^x(1-y)=y$

$0<e^x<e^x+1$ より，$0<y<1$

よって，$e^x=\dfrac{y}{1-y}$　すなわち　$x=\log\dfrac{y}{1-y}$

ゆえに，$y=g(x)=\log\dfrac{x}{1-x}$　$(0<x<1)$

(2)

$y=f(x)$ の逆関数は $x=g(y)$ であり，

$x=g(y)$ の x を y に，y を x にあらためたのが $y=g(x)$ である。

したがって，グラフ $y=f(x)$ と同一 xy 座標平面上で考えると，

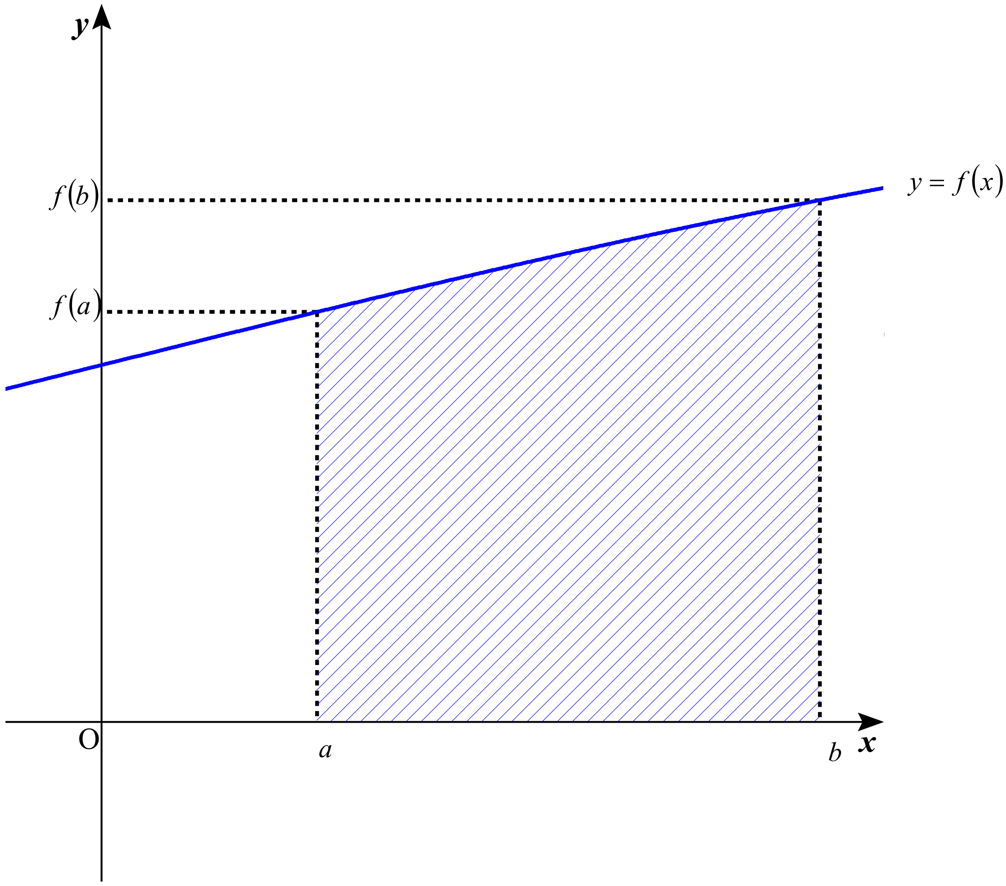
$\int_a^b f(x)dx+\int_{f(b)}^{f(a)} g(x)dx$ は $\int_a^b f(x)dx+\int_{f(a)}^{f(b)} g(y)dy$ となる。

ここで，$g(y)=x$，$\dfrac{dy}{dx}=f'(x)\Leftrightarrow dy=f'(x)dx$，$y=f(b)\Leftrightarrow x=b,\ y=f(a)\Leftrightarrow x=a$ より，

$$
\begin{aligned}
\int_{f(a)}^{f(b)} g(y)dy &= \int_a^b xf'(x)dx \\
&= [xf(x)]_a^b-\int_a^b f(x)dx \\
&= bf(b)-af(a)-\{f(b)-f(a)\}
\end{aligned}
$$

よって，

$$
\begin{aligned}
\int_a^b f(x)dx+\int_{f(a)}^{f(b)} g(x)dx &= \int_a^b f(x)dx+\int_{f(a)}^{f(b)} g(y)dy \\
&= f(b)-f(a)+bf(b)-af(a)-\{f(b)-f(a)\} \\
&= bf(b)-af(a)
\end{aligned}
$$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$y=f(x)$
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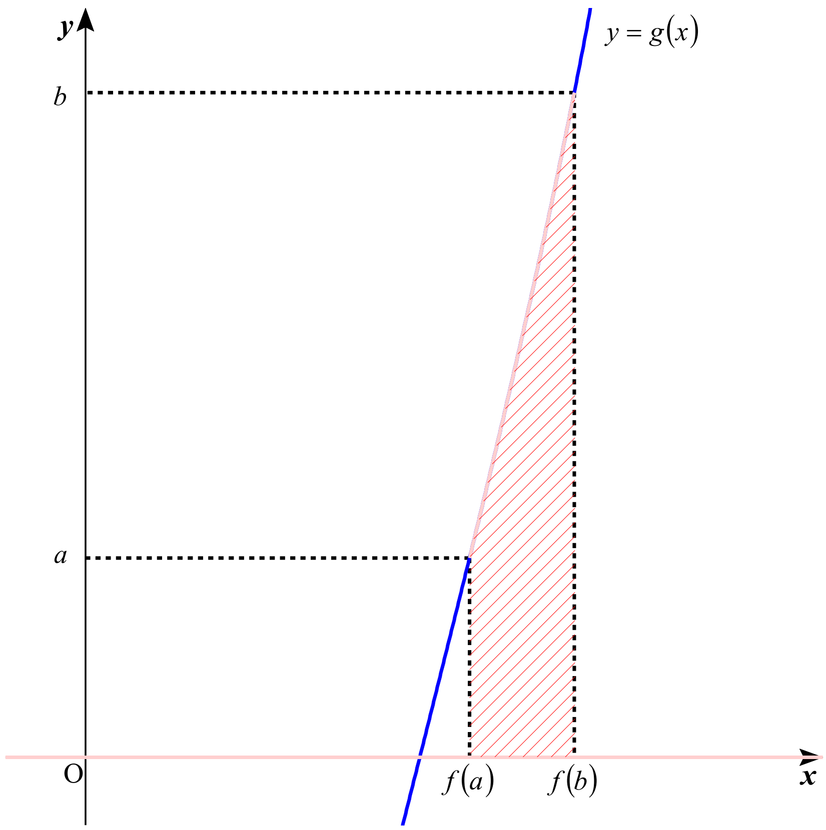
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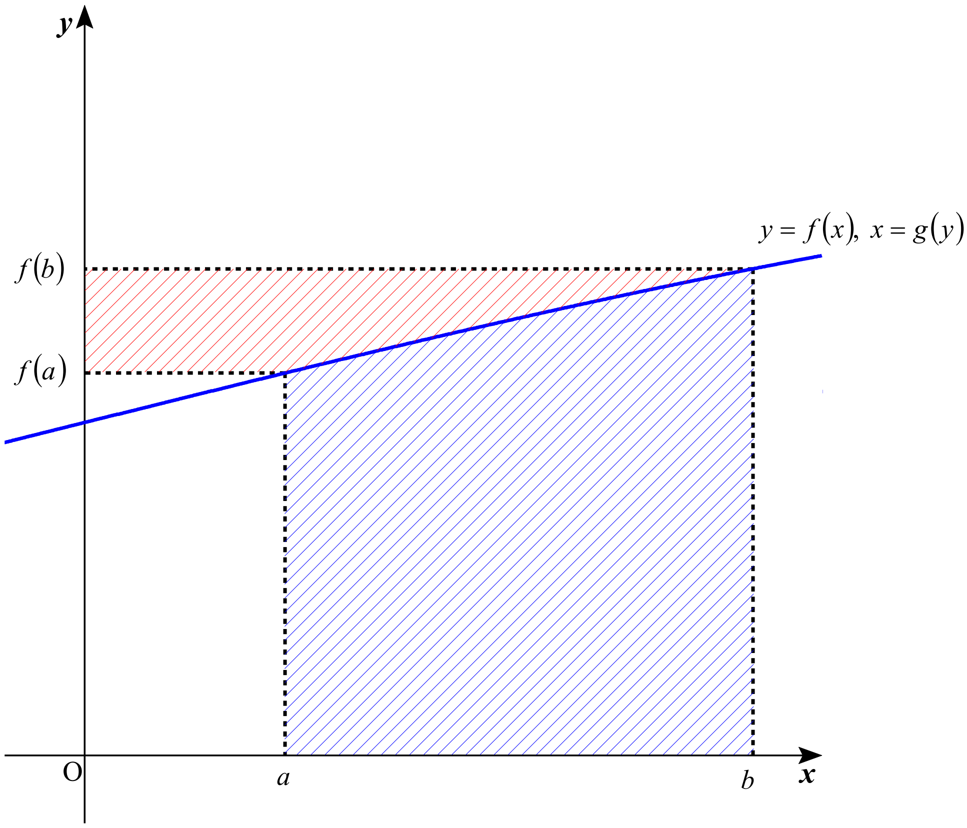
y
f(b)
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y = f(x), x = g(y)
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